Math 206B Lecture 12 Notes

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1 RSK: The (final) Final Chapter

1.1 Properties of generalized RSK

Theorem 1.1. $\Phi : P_\nu(\overline{d}) \to Q_\nu(\overline{d})$ *is*

1. *piecewise linear,*
2. *volume-preserving,*
3. *continuous,*
4. $\Phi : P_\nu \cap \mathbb{Z}^K \to Q_\nu \cap \mathbb{Z}^K$.

We had $\Phi = \xi_{(r_n,s_n)} \circ \xi_{(r_{n-1},s_{n-1})} \circ \cdots$, where $\xi_{(r_i,s_i)}$ is the PL map corresponding to removal of the $(n+1-i)$-th square in $A \in \mathrm{SYT}(\nu)$. Last time, we showed that Φ is well-defined.

Proof. Invertibility of Φ follows from construction, since we can reverse the process.

To prove that Φ is piecewise linear, let ξ send $f(i,j) \mapsto \max\{f(i-1,j), f((i,j-1)\} + \min\{f(i+1,j), f(i,j+1)\} - f(i,j)$. Then ξ sends $f \mapsto \overline{f}$, so this is also volume preserving, as it has determinant ± 1.

We now have to show that $\Phi : P_\nu(\overline{d}) \to Q_\nu(\overline{d})$; that is, if $M \in P_\nu(\overline{d})$ and $\alpha_c(M) = d_c$, then $\Phi(M) \in Q_\nu(\overline{d})$, and $\beta_c(\Phi(M)) = d_c$ for all c. We also need that if $M = [m_{i,j}]$ and $\Phi(M) = [m'_{i,j}]$, then if $m_{i,j} \geq 0$ and $m_{i,j} \leq m_{i,j+1}, m_{i+1,j}$, then $m'_{i,j} \geq 0$. Let's show this 2nd part. We only need to understand this for when we change a corner:

$$\overline{f}(r,s) = f(r,s) - \max\{f(r-1,s), f(r,s-1)\} \geq 0.$$

To show that $\Phi : P_\nu(\overline{d}) \to Q_\nu(\overline{d})$, proceed by induction on each step. Suppose $\xi_{r,s}$ sends $M \mapsto \overline{M}$, where $\overline{M}$ has share $\nu - (r,s)$. We only alter boxes on the diagonal, so $\alpha_c(M) = \alpha_c(\overline{M})$ for all $c \neq r-s$. Now let $u = r-s$. Then

$$\alpha_u(\overline{M}) = \alpha_{u+1}(M) + \alpha_{u-1}(M) - \alpha_u(M) + m'_{r,s}.$$

We also get, using inclusion-exclusion, that

$$\begin{aligned}\beta_c(M') &= \beta_{c+1}(\Phi(M)) + \beta_{c-1}(\Phi(M)) - \beta_c(\Phi(M) - (r,s)) + m'_{r,s} \\ &= \alpha_{c+1}(M) + \alpha_{c-1}(M) - \alpha_{c-1}(M) + m'_{r,s} \\ &= \alpha_u(\Phi(M)).\end{aligned}$$

Comparing these two equalities proves the property by induction. □

1.2 Maximal sum over a path

Proposition 1.1. *Let (r,s) be a corner of ν. Then $m_{r,s}$ is equal to the maximal sum over a path from $(1,1)$ to (r,s) in M'.*

Proof. Look at the formula

$$m'_{r,s} = \overline{f}(r,s) = f(r,s) - \max\{f(r-1,s), f(r,s-1)\}.$$

Then we can prove this by induction. □

When ν is a square, this is the corresponding property for RSK.

What is the moral here? We generalized RSK so far that we can prove all these properties from just two equations involving α and β. Later, we will approach RSK from a different angle, involving f^λ.